HW1 solution

1.1.2. Plugging in the given values of x, y and z gives $a+2b-c=3,\ \ a-2-c=1,\ \ 1+2b+c=2.$ Solving this system yields $a=4, b=0$, and $c=1$.

♡ 1.2.31.

(a) We need AB and BA to have the same size, and so this follows from Exercise 1.2.13.

(b) $AB - BA = \mathrm{O}$ if and only if $AB = BA$.

(c) (*i*) $\begin{pmatrix} -1 & 2 \\ 6 & 1 \end{pmatrix}$, (*ii*) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, (*iii*) $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$;

(d) (*i*)
$$\begin{aligned}[\,cA+dB, C\,] &= (cA+dB)C - C(cA+dB) \\ &= c(AC - CA) + d(BC - CB) = c[\,A,B\,] + d[\,B,C\,], \\ [\,A, cB+dC\,] &= A(cB+dC) - (cB+dC)A \\ &= c(AB - BA) + d(AC - CA) = c[\,A,B\,] + d[\,A,C\,]. \end{aligned}$$

(*ii*) $[\,A,B\,] = AB - BA = -\,(BA - AB) = -\,[\,B,A\,]$.

(*iii*)
$$\begin{aligned} \big[[\,A,B\,],C\big] &= (AB-BA)C - C(AB-BA) = ABC - BAC - CAB + CBA, \\ \big[[\,C,A\,],B\big] &= (CA-AC)B - B(CA-AB) = CAB - ACB - BCA + BAC, \\ \big[[\,B,C\,],A\big] &= (BC-CB)A - A(BC-CB) = BCA - CBA - ABC + ACB. \end{aligned}$$
Summing the three expressions produces O.

◇ 1.2.32. (a) (*i*) 4, (*ii*) 0, (*b*) $\operatorname{tr}(A+B) = \sum_{i=1}^{n}(a_{ii}+b_{ii}) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = \operatorname{tr} A + \operatorname{tr} B$.

(*c*) The diagonal entries of AB are $\sum_{j=1}^{n} a_{ij} b_{ji}$, so $\operatorname{tr}(AB) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} b_{ji}$; the diagonal entries of BA are $\sum_{i=1}^{n} b_{ji} a_{ij}$, so $\operatorname{tr}(BA) = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ji} a_{ij}$. These double summations are clearly equal. (*d*) $\operatorname{tr} C = \operatorname{tr}(AB - BA) = \operatorname{tr} AB - \operatorname{tr} BA = 0$ by part (a).

(*e*) Yes, by the same proof.

◇ 1.3.12.

(a) Set $l_{ij} = \begin{cases} a_{ij}, & i > j, \\ 0, & i \le j, \end{cases}$ $u_{ij} = \begin{cases} a_{ij}, & i < j, \\ 0, & i \ge j, \end{cases}$ $d_{ij} = \begin{cases} a_{ij}, & i = j, \\ 0, & i \ne j. \end{cases}$

(b) $L = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, $U = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

♢ 1.3.13.

(a) By direct computation, $A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and so $A^3 = \mathrm{O}$.

(b) Let A have size $n \times n$. By assumption, $a_{ij} = 0$ whenever $i > j - 1$. By induction, one proves that the (i,j) entries of A^k are all zero whenever $i > j - k$. Indeed, to compute the (i,j) entry of $A^{k+1} = A\,A^k$ you multiply the i^{th} row of A, whose first i entries are 0,

by the j^{th} column of A^k, whose first $j - k - 1$ entries are non-zero, and all the rest are zero, according to the induction hypothesis; therefore, if $i > j - k - 1$, every term in the sum producing this entry is 0, and the induction is complete. In particular, for $k = n$, every entry of A^k is zero, and so $A^n = \mathrm{O}$.

(c) The matrix $A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ has $A^2 = \mathrm{O}$.

1.5.20.

(a) $B\,A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(b) $A\,X = \mathrm{I}$ does not have a solution. Indeed, the first column of this matrix equation is the linear system $\begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, which has no solutions since $x - y = 1$, $y = 0$, and $x + y = 0$ are incompatible.

(c) Yes: for instance, $B = \begin{pmatrix} 2 & 3 & -1 \\ -1 & -1 & 1 \end{pmatrix}$. More generally, $B\,A = \mathrm{I}$ if and only if $B = \begin{pmatrix} 1 - z & 1 - 2z & z \\ -w & 1 - 2w & w \end{pmatrix}$, where z, w are arbitrary.

1.6.30.

(a) Let $S = \frac{1}{2}(A + A^T)$, $J = \frac{1}{2}(A - A^T)$. Then $S^T = S$, $J^T = -J$, and $A = S + J$.

(b) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & \frac{5}{2} \\ \frac{5}{2} & 4 \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$; $\quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{pmatrix} + \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}$.

♢ 1.9.21. Using the LU factorizations established in Exercise 1.3.25:

(a) $\det \begin{pmatrix} 1 & 1 \\ t_1 & t_2 \end{pmatrix} = t_2 - t_1$, (b) $\det \begin{pmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \\ t_1^2 & t_2^2 & t_3^2 \end{pmatrix} = (t_2 - t_1)(t_3 - t_1)(t_3 - t_2)$,

(c) $\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ t_1 & t_2 & t_3 & t_4 \\ t_1^2 & t_2^2 & t_3^2 & t_4^2 \\ t_1^3 & t_2^3 & t_3^3 & t_4^3 \end{pmatrix} = (t_2 - t_1)(t_3 - t_1)(t_3 - t_2)(t_4 - t_1)(t_4 - t_2)(t_4 - t_3)$.

The general formula is found in Exercise 4.4.29.